MATH 54 - MOCK MIDTERM 2 - SOLUTIONS

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1. *(50 points, 5 pts each)*

Label the following statements as **T** or **F**.

Make sure to **JUSTIFY YOUR ANSWERS!!!** You may use any facts from the book or from lecture.

(a) If A and B are square matrices, then $det(A+B) = det(A) + det(B)$.

FALSE

For example, take $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.

Then $det(A) = 1$, $det(B) = 1$, but $det(A+B) = det(O) = 0$ (where O is the zero-matrix).

(b) If $\mathcal{A} = \{\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}\}$ and $\mathcal{D} = \{\mathbf{d_1}, \mathbf{d_2}, \mathbf{d_3}\}$ are bases for V, and P is the matrix whose ith column is $[\mathbf{d_i}]_{\mathcal{A}}$, then for all $\mathbf{x}$ in V, we have $[\mathbf{x}]_{\mathcal{D}} = P\,[\mathbf{x}]_{\mathcal{A}}$

FALSE

First of all, $P = \begin{bmatrix} [\mathbf{d_1}]_{\mathcal{A}} & [\mathbf{d_2}]_{\mathcal{A}} & [\mathbf{d_3}]_{\mathcal{A}} \end{bmatrix} = \underset{\mathcal{A} \leftarrow \mathcal{D}}{P}$ (remember, you always evaluate with respect to the new, cool basis, here it is $\mathcal{A}$), so we should have:

$$[\mathbf{x}]_{\mathcal{A}} = \underset{\mathcal{A} \leftarrow \mathcal{D}}{P}\,[\mathbf{x}]_{\mathcal{D}} = P\,[\mathbf{x}]_{\mathcal{D}}$$

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And not the opposite!

(c) If $Nul(A) = \{\mathbf{0}\}$, then A is invertible.

FALSE

Don't worry, this got me too! This statement *is* true if A is **SQUARE** ! But if A is not square, this statement is never true!

For example, let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then $Nul(A) = \{\mathbf{0}\}$, but A is not invertible, because it is not square.

(d) A 3×3 matrix A with only one eigenvalue cannot be diagonalizable

SUPER FALSE!!!!!!!!!!

Remember that to check if a matrix is not diagonalizable, you really have to look at the eigenvectors!

For example, $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ has only eigenvalue 2, but is diagonalizable (it's diagonal!). Or you can choose A to be the O matrix, or the identity matrix, this also works!

(e) $\mathbb{R}^2$ is a subspace of $\mathbb{R}^3$

FALSE!

$\mathbb{R}^2$ is not even a *subset* of $\mathbb{R}^3$!!! Don't confuse this with $\left\{ \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \mid x, y \in \mathbb{R} \right\}$, which *is* a subspace of $\mathbb{R}^3$ and very similar to $\mathbb{R}^2$ (but not <u>exactly</u> the same)

(f) If W is a subspace of V and $\mathcal{B}$ is a basis for V, then some subset of $\mathcal{B}$ is a basis for W.

FALSE

This is also very tricky (this got me too :)), because the 'opposite' statement does hold, namely if $\mathcal{B}$ is a basis for W, you can always complete $\mathcal{B}$ to become a basis of V (this is the 'basis extension theorem').

As a counterexample, take $V = \mathbb{R}^3$, $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$, and $W = Span \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ (a line in $\mathbb{R}^3$).

If the statement was true, then one of the vectors in $\mathcal{B}$ would be a basis for W, but this is bogus.

(g) If $\mathbf{v_1}$ and $\mathbf{v_2}$ are 2 eigenvectors of A corresponding to 2 **different** eigenvalues λ_1 and λ_2, then $\mathbf{v_1}$ and $\mathbf{v_2}$ are linearly independent!

TRUE (finally!)

Note: The proof is a bit complicated, but I've seen this on a past exam! I think at that point, the professor wanted to get revenge on his students for not coming to lecture!

Remember that eigenvectors have to be nonzero!

Now, assume $a\mathbf{v_1} + b\mathbf{v_2} = \mathbf{0}$.

Then apply A to this to get:

$$A(a\mathbf{v_1} + b\mathbf{v_2}) = A(\mathbf{0}) = \mathbf{0}$$

That is:

$$aA(\mathbf{v_1}) + bA(\mathbf{v_2}) = \mathbf{0}$$

$$a\lambda_1\mathbf{v_1} + b\lambda_2\mathbf{v_2} = \mathbf{0}$$

However, we can also multiply the original equation by λ_1 to get:

$$a\lambda_1\mathbf{v_1} + b\lambda_1\mathbf{v_2} = \mathbf{0}$$

Subtracting this equation from the one preceding it, we get:

$$b(\lambda_1 - \lambda_2)\mathbf{v_2} = \mathbf{0}$$

So

$$b(\lambda_1 - \lambda_2) = \mathbf{0}$$

But $\lambda_1 \neq \lambda_2$, so $\lambda_1 - \lambda_2 \neq 0$, hence we get $b = 0$.

But going back to the first equation, we get:

$$a\mathbf{v_1} = \mathbf{0}$$

So $a = 0$.

Hence $a = b = 0$, and we're done!

(h) If a matrix A has orthogonal columns, then it is an orthogonal matrix.

FALSE

Remember that an ortho**gonal** matrix has to have ortho**normal** columns!

(i) For every subspace W and every vector $\mathbf{y}$, $\mathbf{y} - Proj_W\mathbf{y}$ is orthogonal to $Proj_W\mathbf{y}$ (proof by picture is ok here)
TRUE

Draw a picture! $Proj_W\mathbf{y}$ is just another name for $\hat{y}$.

(j) If $\mathbf{y}$ is already in W, then $Proj_W \mathbf{y} = \mathbf{y}$

TRUE

Again, draw a picture!

If you want a more mathematical proof, here it is:

Let $\mathcal{B} = \{\mathbf{w_1}, \cdots \mathbf{w_p}\}$ be an orthogonal basis for W ($p = Dim(W)$).

Then $y = \left(\frac{\mathbf{y} \cdot \mathbf{w_1}}{\mathbf{w_1} \cdot \mathbf{w_1}}\right) \mathbf{w_1} + \cdots + \left(\frac{\mathbf{y} \cdot \mathbf{w_p}}{\mathbf{w_p} \cdot \mathbf{w_p}}\right) \mathbf{w_p}$.

But then, by definition of $Proj_W \mathbf{y} = \hat{\mathbf{y}}$, we get:

$$\hat{y} = \left(\frac{\mathbf{y} \cdot \mathbf{w_1}}{\mathbf{w_1} \cdot \mathbf{w_1}}\right) \mathbf{w_1} + \cdots + \left(\frac{\mathbf{y} \cdot \mathbf{w_p}}{\mathbf{w_p} \cdot \mathbf{w_p}}\right) \mathbf{w_p} = y$$

So $\hat{\mathbf{y}} = \mathbf{y}$ in this case.

2. *(20 points)* Find a diagonal matrix D and an invertible matrix P such that $A = PDP^{-1}$, where:

$$A = \begin{bmatrix} 7 & -6 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 7 \end{bmatrix}$$

<u>Eigenvalues:</u> $det(A - \lambda I) = 0$ (expanding along last column), which gives $(\lambda - 1)(\lambda - 7)^2 = 0$, so $\boxed{\lambda = 1, 7}$

$\underline{\lambda = 1}$

$$Nul(A - I) = Nul\left(\begin{bmatrix} 6 & -6 & 0 \\ 0 & 0 & 0 \\ 0 & 3 & 6 \end{bmatrix}\right) = Nul\left(\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}\right) = Span\left\{\begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}\right\}$$

$\underline{\lambda = 7}$

$$Nul(A - 7I) = Nul\left(\begin{bmatrix} 0 & -6 & 0 \\ 0 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix}\right) = Nul\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = Span\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$$

Hence:

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}, P = \begin{bmatrix} -2 & 1 & 0 \\ -2 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

3. *(20 points)* Use the Gram-Schmidt process to obtain an orthonormal basis of $W = Span\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$, where:

$$\mathbf{v_1} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 3 \end{bmatrix}, \mathbf{v_2} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \mathbf{v_3} = \begin{bmatrix} 6 \\ -3 \\ 1 \\ 11 \end{bmatrix}$$

First of all, $\{\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}\}$ is an orthogonal basis for W, where:

$$\mathbf{w_1} = \mathbf{v_1} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 3 \end{bmatrix}$$

$$\mathbf{w_2} = \mathbf{v_2} - \left(\frac{\mathbf{v_2} \cdot \mathbf{w_1}}{\mathbf{w_1} \cdot \mathbf{w_1}} \right) \mathbf{w_1} = \mathbf{v_2} - \frac{7}{14}\mathbf{w_1} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 2 \\ 1 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{3}{2} \\ 0 \\ -\frac{1}{2} \end{bmatrix} \sim \begin{bmatrix} 0 \\ 3 \\ 0 \\ -1 \end{bmatrix}$$

$$\begin{aligned} \mathbf{w_3} = \mathbf{v_3} - \left(\frac{\mathbf{v_3} \cdot \mathbf{w_1}}{\mathbf{w_1} \cdot \mathbf{w_1}} \right) \mathbf{w_1} - \left(\frac{\mathbf{v_3} \cdot \mathbf{w_2}}{\mathbf{w_2} \cdot \mathbf{w_2}} \right) \mathbf{w_2} &= \mathbf{v_3} - \frac{42}{14}\mathbf{w_1} - \frac{-20}{10}\mathbf{w_2} \\ &= \mathbf{v_3} - 3\mathbf{w_1} + 2\mathbf{w_2} = \begin{bmatrix} 6 \\ -3 \\ 1 \\ 11 \end{bmatrix} - 3 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 3 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 3 \\ 0 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \end{aligned}$$

Finally, an orthonormal basis for W is $\{\mathbf{w_1}', \mathbf{w_2}', \mathbf{w_3}'\}$, where:

$$\mathbf{w_1}' = \frac{\mathbf{w_1}}{\|\mathbf{w_1}\|} = \frac{1}{\sqrt{14}} \begin{bmatrix} 2 \\ 1 \\ 0 \\ 3 \end{bmatrix}, \mathbf{w_2}' = \frac{\mathbf{w_2}}{\|\mathbf{w_2}\|} = \frac{1}{\sqrt{10}} \begin{bmatrix} 0 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \mathbf{w_3}' = \frac{\mathbf{w_3}}{\|\mathbf{w_3}\|} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

4. *(10 points)* Find the determinant of the following matrix A:

$$A = \begin{bmatrix} 1 & 42 & 536 & 789 & 4201 & 123456789 \\ 0 & 1 & 2011 & 2012 & \pi m & \text{Dolphin} \\ 0 & 0 & 2 & 0 & 4 & 5 \\ 0 & 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 & 2 & -1 \end{bmatrix}$$

Note: The answer may surprise you :)

First of all, expanding along the first column, and then along the first column again, we get that:

$$det(A) = \begin{vmatrix} 2 & 0 & 4 & 5 \\ 0 & 0 & 3 & 1 \\ 0 & 1 & 0 & 0 \\ 4 & 0 & 2 & -1 \end{vmatrix}$$

Now expanding along the 3rd row (or the second column), we get:

$$det(A) = -\begin{vmatrix} 2 & 4 & 5 \\ 0 & 3 & 1 \\ 4 & 2 & -1 \end{vmatrix}$$

Note: Careful about the signs!

Finally, expanding along the second row (or first column), we get:

$$det(A) = -\left(3\begin{vmatrix} 2 & 5 \\ 4 & -1 \end{vmatrix} - \begin{vmatrix} 2 & 4 \\ 4 & 2 \end{vmatrix}\right) = -\left((3)(-22) + 12\right) = 54$$

NO WAY!!! I know, right? I did not expect that at all! :D

Note: Here's a smarter way to evaluate $det(A)$ (courtesy Rongchang Lei): Just row-reduce!

$$
\begin{aligned}
det(A) &= \begin{vmatrix} 2 & 0 & 4 & 5 \\ 0 & 0 & 3 & 1 \\ 0 & 1 & 0 & 0 \\ 4 & 0 & 2 & -1 \end{vmatrix} \\
&= \begin{vmatrix} 2 & 0 & 4 & 5 \\ 0 & 0 & 3 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -6 & -11 \end{vmatrix} \qquad (R_4 - 2R_1) \\
&= - \begin{vmatrix} 2 & 0 & 4 & 5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & -6 & -11 \end{vmatrix} \qquad (R_2 \leftrightarrow R_3) \\
&= - \begin{vmatrix} 2 & 0 & 4 & 5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & -9 \end{vmatrix} \qquad (R_4 - 2R_3) \\
&= -(2)(1)(3)(-9) \qquad \text{upper triangular matrix} \\
&= 54
\end{aligned}
$$

5. *(10 points)* Find a least squares solution to the following system $A\mathbf{x} = \mathbf{b}$, where:

$$A = \begin{bmatrix} 2 & 0 \\ 1 & -1 \\ -1 & 2 \\ 0 & 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$A^T A \hat{\mathbf{x}} = A^T \mathbf{b}$$

$$\begin{bmatrix} 2 & 1 & -1 & 0 \\ 0 & -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 1 & -1 \\ -1 & 2 \\ 0 & 1 \end{bmatrix} \hat{\mathbf{x}} = \begin{bmatrix} 2 & 1 & -1 & 0 \\ 0 & -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 6 & -3 \\ -3 & 6 \end{bmatrix} \hat{\mathbf{x}} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

Which gives:

$$\hat{\mathbf{x}} = \begin{bmatrix} 6 & -3 \\ -3 & 6 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{5}{9} \\ \frac{1}{9} \end{bmatrix}$$

6. *(10 points)* Define $T : P_3 \to P_3$ by:

$$T(p(t)) = tp''(t) - 2p'(t)$$

Find the matrix A of T relative to the basis $\mathcal{B} = \{1, t, t^2, t^3\}$ of P_3

First calculate:

- $T(1) = t(0) - 2(0) = 0$
- $T(t) = t(0) - 2(1) = -2$
- $T(t^2) = t(2) - 2(2t) = -2t$
- $T(t^3) = t(6t) - 2(3t^2) = 6t^2 - 6t^2 = 0$

Now evaluate all those vectors with respect to $\mathcal{B}$:

- $[T(1)]_{\mathcal{B}} = [0]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$
- $[T(t)]_{\mathcal{B}} = [-2]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$
- $[T(t^2)]_{\mathcal{B}} = [-2t]_{\mathcal{B}} = \begin{bmatrix} 0 \\ -2 \\ 0 \\ 0 \end{bmatrix}$
- $[T(t^3)]_{\mathcal{B}} = [0]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

Putting everything together, we get that the matrix of T relative to $\mathcal{B}$ is:

$$A = \begin{bmatrix} 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

7. *(15 points)* Let $\mathcal{B} = \left\{ \begin{bmatrix} 7 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\}, \mathcal{C} = \left\{ \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \end{bmatrix} \right\}$.

(a) Find the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$

We want to find $\mathcal{C} \overset{P}{\leftarrow} \mathcal{B}$.

$$\left[\ \mathcal{C} \mid \mathcal{B}\ \right] \to \left[\begin{array}{cc|cc} 4 & 5 & 7 & 2 \\ 1 & 2 & -2 & -1 \end{array}\right] \to \left[\begin{array}{cc|cc} 4 & 5 & 7 & 2 \\ 0 & -3 & 15 & 6 \end{array}\right] \to \left[\begin{array}{cc|cc} 4 & 5 & 7 & 2 \\ 0 & 1 & -5 & -2 \end{array}\right]$$
$$\to \left[\begin{array}{cc|cc} 4 & 0 & 32 & 12 \\ 0 & 1 & -5 & -2 \end{array}\right]$$
$$\to \left[\begin{array}{cc|cc} 1 & 0 & 8 & 3 \\ 0 & 1 & -5 & -2 \end{array}\right]$$

Hence:

$$\mathcal{C} \overset{P}{\leftarrow} \mathcal{B} = \begin{bmatrix} 8 & 3 \\ -5 & -2 \end{bmatrix}$$

(b) Find $[\mathbf{x}]_{\mathcal{C}}$, where $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$

We have:

$$[\mathbf{x}]_{\mathcal{C}} = \mathcal{C} \overset{P}{\leftarrow} \mathcal{B}\ [\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 8 & 3 \\ -5 & -2 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 47 \\ -30 \end{bmatrix}$$

8. *(10 points)* Find the orthogonal projection of t^2 onto the subspace W spanned by $\{1, t\}$, with respect to the following inner product:

$$\langle p, q\rangle = \int_{-1}^{1} p(t)q(t)dt$$

Let $p_1(t) = 1$, $p_2(t) = t$, $p_3(t) = t^2$, then:

$$\hat{p_3} = \left(\frac{\langle p_3, p_1\rangle}{\langle p_1, p_1\rangle}\right) p_1 + \left(\frac{\langle p_3, p_2\rangle}{\langle p_2, p_2\rangle}\right) p_2 = \left(\frac{\int_{-1}^{1} t^2 dt}{\int_{-1}^{1} dt}\right)(1) + \left(\frac{\int_{-1}^{1} t^3 dt}{\int_{-1}^{1} t^2 dt}\right)(t) = \left(\frac{\frac{2}{3}}{2}\right)(1) + \left(\frac{0}{\frac{2}{3}}\right) t = \frac{1}{3}$$

$$\hat{p_3}(t) = \frac{1}{3}$$

9. *(20 points, 10 points each)*

(a) Find a basis for $Row(A)$ and $Col(A)$, where:

$$A = \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ -2 & 3 & -3 & -3 & -4 \\ 4 & -6 & 9 & 5 & 0 \\ -2 & 3 & 3 & -4 & 1 \end{bmatrix}$$

If you row-reduce A, you get that:

$$A \sim \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ 0 & 0 & 3 & -1 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Note: You could have row-reduced it further, but no need!

Notice that the pivots are in the all 4 rows and the 1st, 3rd, 4th, and 5th column respectively, hence:

<u>Basis for Row(A):</u>

$$\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ -3 \\ 6 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 3 \end{bmatrix} \right\}$$

<u>Basis for Col(A):</u>

$$\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ -2 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} 6 \\ -3 \\ 9 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 5 \\ -4 \end{bmatrix}, \begin{bmatrix} 5 \\ -4 \\ 0 \\ 1 \end{bmatrix} \right\}$$

(b) What is $Rank(A)$? What is $Dim(Nul(A))$?

$Rank(A) = 4$ (number of pivots)

$Dim(Nul(A)) = 5 - Rank(A) = 5 - 4 = 1$ (by Rank-Nullity theorem)

10. *(15 points)*

(a) Find an invertible matrix P and a matrix C of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ such that $A = PCP^{-1}$, where:

$$A = \begin{bmatrix} 2 & -2 \\ 1 & 0 \end{bmatrix}$$

<u>Eigenvalues:</u>

The characteristic polynomial of A is: $det(A - \lambda I) = (\lambda - 2)(\lambda) + 2 = \lambda^2 - 2\lambda + 2 = 0$ iff $\lambda = 1 \pm i$

<u>Eigenspace for $\lambda = 1 - i$</u>

$$Nul(A-(1-i)I) = Nul\left(\begin{bmatrix} 1+i & -2 \\ 1 & -1+i \end{bmatrix}\right) = Nul\left(\begin{bmatrix} 1 & -1+i \\ 0 & 0 \end{bmatrix}\right) = Span\left\{\begin{bmatrix} 1-i \\ 1 \end{bmatrix}\right\}$$

So an eigenvector corresponding to $\lambda = 1 - i$ is $\mathbf{v} = \begin{bmatrix} 1-i \\ 1 \end{bmatrix}$

<u>Finding P and C:</u>

First of all, for P, we have:

$$Re(\mathbf{v}) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, Im(\mathbf{v}) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

Hence: $P = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$.

As for C, we have $Re(\lambda) = 1$, $Im(\lambda) = -1$. Now remember that you put those values on the first **ROW** of C, and you get:

$$C = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

(Remember that the diagonal entries of C are equal and the nondiagonal ones are opposite of each other)

(b) Write C as a composition of a rotation and a scaling.

C is a rotation by ϕ followed by a scaling r.

r is given by: $r = \sqrt{det(C)} = \sqrt{2}$, hence:

$$C = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

If you compare this latter matrix with the rotation matrix $\begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix}$, you should realize that $\phi = \frac{\pi}{4}$.

Hence C is a rotation by $\boxed{\phi = \frac{\pi}{4}}$ followed by a scaling $\boxed{r = \sqrt{2}}$.